RIGOROUS LOWER BOUND ON THE COMPRESSIBILITY OF A CLASSICAL SYSTEM

J. GINIBRE
Laboratoire de Physique Théorique et Hautes Energies, Orsay*

Received 7 January 1967

A simple inequality is proved, from which a rigorous lower bound on the compressibility of a classical system with purely repulsive or hard core interaction follows immediately.

In this note we prove the following simple inequality: let p be a probability measure on the positive integers. Let $p_n = p_n/n!$ be the probability of integer n. Suppose that the second moment of p exists and that for all $n \geqslant 0$:

$$(P_{n+2}/P_{n+1}) \geqslant (P_{n+1}/P_n) - A, \quad \text{where } A > -1. \quad \text{(a)}$$

If some $P_n = 0$, it is to be understood that $P_m = 0$ for all $m \geqslant n$. Then:

$$\frac{\langle (n - \langle n \rangle)^2 \rangle}{\langle n \rangle} \geqslant \frac{1}{1+A} . \quad (1)$$

Equality in (1) is obtained:

if $A > 0$, for $P_n = \frac{N!}{(N-n)!} \frac{A^n}{(1+A)^N}$ for $n \leqslant N$,

(N arbitrary integer $\geqslant 1$) and zero otherwise.

if $A = 0$, for $P_n = \alpha^n e^{-\alpha}$ with α real, $\alpha > 0$.

if $-1 < A < 0$, for $P_n =$

$= (-A)^n (1+A)^\alpha \alpha(\alpha+1)\dots(\alpha+n-1)$ with α real, $\alpha > 0$.

Proof: From Schwarz' inequality:

$$\{\langle n \rangle (1+A)\}^2 = \left\{\sum_0^\infty \frac{1}{n!} P_n\left(\frac{P_{n+1}}{P_n} + nA\right)\right\}^2 \leqslant$$

$$\leqslant \sum_0^\infty \frac{1}{n!} P_n\left(\frac{P_{n+1}}{P_n} + nA\right)^2 . \quad (2)$$

expanding the right hand side and using (a), one gets (1). The cases of equality in (1) are obtained easily.

As an application, consider a system of classical particles, enclosed in a box of volume V, interacting through a 2 body potential φ; the system is described in the grand canonical formalism. Let z be the fugacity, Z the grand partition function, T the absolute temperature, and $\beta = (kT)^{-1}$. This defines a probability p on the integers by:

$$P_n = Z^{-1} z^n \int dx_1 \dots dx_n \exp[-\beta \sum_{i<j} \varphi(x_i - x_j)]. \quad (3)$$

For purely repulsive potentials and for hard core potentials, i.e. potentials such that (1): $\varphi(x-y) = +\infty$ for $|x-y| < a$, (2): for any finite sequence of $n+1$ points $x_0, \dots, x_n$ such that $|x_i - x_j| \geqslant a$ for all i and j,

$$\sum_{i=1}^n \varphi(x_0 - x_i) \geqslant -B,$$

where B is a real (positive) constant, one can prove easily [1,2] under the additional assumption

$$\int (1 - \exp[-\beta \varphi_+(x)])\, dx < \infty, \quad \text{(b)}$$

where $\varphi_+(x) = \max(\varphi(x), 0)$, that condition (a) is satisfied, with:

$$A = zD = z\, e^{\beta B} \int (1 - \exp[-\beta \varphi_+(x)])\, dx \quad (4)$$

(For repulsive potentials, $B = 0$ and $\varphi_+ = \varphi$.) Therefore, for all positive z and V:

$$\chi = \frac{1}{\rho} \frac{d\rho}{dP} = \frac{\beta}{\rho} \frac{\langle (n - \langle n \rangle)^2 \rangle}{\langle n \rangle} \geqslant \frac{\beta}{\rho} \frac{1}{1+zD}$$

where $P = \ln Z/V$ is the pressure, $\rho = \langle n \rangle / V$ the

* Postal address: Laboratoire de Physique Théorique et Hautes Energies, Bât. 211, Faculté des Sciences, 91 - Orsay (France).

density, and χ the compressibility. For repulsive or hard core potentials satisfying a weak tempering condition which is closely related to (b) and slightly stronger, the thermodynamic limit exists [3]. Inequality (5) still holds in the limit as a Lipschitz condition on the pressure. This proves directly in the grand canonical formalism the continuity of the pressure as a function of the density (proofs have already been given in the canonical formalism under various assumptions [1,2,4]) and gives a lower bound on the compressibility.

1. O. Penrose, private communication to D. Ruelle.
2. F.A. Berezin, R.L. Dobrushin, R.A. Minlos, A.Ya. Povzner and Ya.G. Sinai, preprint.
3. M. Fisher, Arch for Ratl. Mech. and An. 17 (1964) 377.
4. D. Ruelle, Helv. Phys. Acta 36 (1963) 183.

* * * * *